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Finite Element Formulation for Thick Sandwich Plates on an Elastic Foundation

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To construct a plate theory for a thick transversely compressible sandwich plate with composite laminated face sheets, the authors make simplifying assumptions regarding distribution of transverse strain components in the thickness direction. It is assumed that the transverse strains (i.e., ε_{xz}, ε_{yz}, and ε_{zz}) do not vary in the thickness direction within the face sheets and the core, but can be different functions of the in-plane coordinates in different sublaminates (the face sheets and the core). An algorithm, which takes account of damage progression in dynamic problems, is incorporated into the computational scheme based on the geometrically nonlinear formulation and is applied to failure analysis of a sandwich plate under ground impact. In the finite element analysis of sandwich plates with small thickness-to-length ratios, the shear locking phenomenon does not occur. The model of the sandwich plate, presented in this paper, does not require many degrees of freedom in the finite element computations and has a wide range of applicability: It can be applied to the sandwich plates with a wide range of ratios of thickness to the in-plane dimensions, with both thin and thick face sheets (as compared to the thickness of the core) and to the sandwich plates with both transversely rigid and transversely compressible face sheets and cores.

Introduction

SANDWICH structures are used in a variety of load-bearing applications. Sandwich plates have a well pronounced zigzag variation of the in-plane displacements in the thickness direction, due to their high thickness-to-length ratios and large difference of values of elastic moduli of the face sheets and the core. Such characteristics of the sandwich plates make it necessary to use a layerwise approach in their analysis, the idea of which is to introduce separate simplifying assumptions regarding the through-thickness variation of displacements, strains, or stresses within each face sheet and the core. Many researchers studied the sandwich plate with thick, lightweight, vertically stiff cores, and thin rigid face sheets, using discrete-layer (or layerwise) models. Most of the layerwise models of such structures are based on the piecewise linear through the thickness in-plane displacement approximations in addition to constant (though-the-thickness) transverse displacements.[1–9]

The modern cores are usually made of plastic foams and nonmetallic honeycombs, like Aramid and Nomex. These cores have properties similar to those used traditionally (for example, metallic honeycombs), but due to their transverse compressibility (i.e., ability of such cores to change height under applied loads) the direct transverse strain ε_{zz} becomes important. Therefore, the models of the sandwich plates with the cores made of plastic foams or nonmetallic honeycombs must not exclude the change of height of the core. Frostig and Baruch[10] developed a theory of a sandwich beam with thin face sheets in which account is taken of transverse compressibility of the core, and the longitudinal displacement in the core varies nonlinearly in the thickness direction. In this theory, the longitudinal displacement in the face sheets varies linearly in the thickness direction, and the transverse displacement of the face sheets does not vary in the thickness direction, that is, the transverse direct strain ε_{zz} in the face sheets is assumed to be equal to zero in the expression for the strain energy. The transverse shear strain ε_{xz} in the face sheets is also considered to be negligibly small in the expression for the strain energy that is used for variational derivation of the differential equations for the unknown functions. The transverse shear stress in the face sheets can be computed by integration of the pointwise equilibrium equations $\sigma_{xx,x} + \sigma_{xz,z} = 0$.

Under certain circumstances, when the face sheets are thick, when the plate is loaded by a concentrated or partially distributed load, or when the plate is on an elastic foundation, taking account of the direct transverse strain ε_{zz} in the face sheets and the transverse shear strain ε_{xz} in the face sheets in the expression for the strain energy allows one to obtain a higher accuracy of the stress computation. Besides, to achieve a high accuracy of stress computation in the thick face sheets, a model for such a plate must assume or lead to the nonlinear through-the-thickness variation of the in-plane displacements, not only in the core, but also in the face sheets.

Construction of a computational scheme that satisfies these requirements can be approached, for example, with the help of the layerwise laminated plate theory of Reddy,[11] which is a generalization of many other displacement-based layerwise theories of laminated plates. In this theory, the displacement field in the kth layer is written as

$$u^{(k)}(x, y, z, t) = \sum_{j=1}^{m} u_j^{(k)}(x, y, t)\phi_j^{(k)}(z)$$

$$v^{(k)}(x, y, z, t) = \sum_{j=1}^{m} v_j^{(k)}(x, y, t)\phi_j^{(k)}(z)$$

$$w^{(k)}(x, y, z, t) = \sum_{j=1}^{n} w_j^{(k)}(x, y, t)\psi_j^{(k)}(z)$$

where $u_j^{(k)}(x, y, t)$, $v_j^{(k)}(x, y, t)$, and $w_j^{(k)}(x, y, t)$ are the unknown functions and $\phi_j^{(k)}(z)$ and $\psi_j^{(k)}(z)$ are chosen to be Lagrange interpolation functions of the thickness coordinate, to provide the required continuity of displacements and discontinuity of the transverse strains across the interface between adjacent thickness subdivisions. This theory allows one to achieve a high accuracy of the transverse stress computation in the composite laminates, but

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for this purpose it requires a large number of thickness subdivisions of the laminate. This leads to a large number of the unknown functions and degrees of freedom in a finite element model. In effect, the finite element model, based on this generalized layerwise laminated plate theory, is equivalent to the three-dimensional finite element model. To reduce the number of the unknown functions in the layerwise model of a laminated plate, one can use the concept of a sublaminate (i.e., make the number of thickness subdivisions less than the number of material layers) and deal with the material properties, averaged through the thickness of a sublaminate. In a model of the sandwich plate, it is natural to choose three sublaminates: the two face sheets and the core. With such a small number of sublaminates, the nature of assumptions on the through-the-thickness variation of displacements can have a large effect on the accuracy of the computed stresses. Besides, the actual through-the-thickness variation of displacements can depend on the character of applied loads and boundary conditions. Therefore, in a layerwise model of the sandwich plate with only three sublaminates, it is desirable to have flexibility in the choice of the functions that represent through-the-thickness variation of displacements. Of course, the Lagrange interpolation polynomials, which represent the thickness variation of the displacements within a sublaminate in the Reddy's[11] layerwise theory, can be chosen to be of any desired degree, but such an increase of the degree of the Lagrange interpolation polynomials leads to the increase of the number of the unknown functions.

In the present paper, we construct a computational scheme for analysis of the sandwich plate in which the simplifying assumptions that lead to a plate-type theory are made with respect to the variation of the transverse strains in the thickness direction of the face sheets and the core of the sandwich plate. The displacements are then obtained by integration of these assumed transverse strains, and the constants of integration are chosen to satisfy the conditions of continuity of the displacements across the borders between the face sheets and the core. In such a method, the required continuity of displacements in the thickness direction is satisfied regardless of the assumed type of through-the-thickness distribution of the transverse strains, and the transverse flexibility of the plate can be taken into account. This leads to a larger number of choices of simplifying assumptions about the variation of strains (and, therefore, displacements) in the thickness direction and, therefore, allows a better adjustment of the computational scheme to the conditions under which the sandwich plate is analyzed by a layerwise method with only three sublaminates (being the face sheets and the core). The transverse stresses are computed by integration of the pointwise equilibrium equations that leads to satisfaction of conditions of continuity of the transverse stresses across the boundaries between the face sheets and the core and satisfaction of stress boundary conditions on the upper and lower surfaces of the plate.

In the present paper, we consider the model based on the simplest of such assumptions that do not ignore in the expression for the strain energy the transverse shear and normal strains in the face sheets. We assume that the transverse strains do not vary in the thickness direction within the face sheets and the core, but can be different functions of the in-plane coordinate in the face sheets and the core. In the postprocess stage of analysis, these first approximations of the transverse strains can be improved by substituting the transverse stresses, obtained by integration of the pointwise equations of motion (Appendix) into the strain–stress relations. These improved values of the transverse strains vary in the thickness direction and are sufficiently accurate as compared to those of the known exact solutions, based on the linear three-dimensional theory.[12] In the theory, discussed in this paper, the transverse displacement, obtained by integration of the assumed transverse normal strain, varies linearly in the thickness direction within a sublaminate. (Therefore, transverse compressibility of the plate is taken into account.) The in-plane displacement, obtained by integration of the assumed transverse shear strains, varies quadratically within the thickness of a sublaminate. The developed theory does not require many degrees of freedom in finite element models, despite its ability to capture the transverse flexibility of the plate and nonlinear through-the-thickness variation of the in-plane displacements.

Three-Dimensional Formulation

The sandwich plate is divided into three conventional layers (sublaminates): the two face sheets and the core. Within each sublaminate, the simplifying assumptions of the plate theory are made separately. In the following text, the superscript k denotes the number of a sublaminate: $k = 1$ denotes the lower face sheet, $k = 2$ denotes the core, and $k = 3$ denotes the upper face sheet (Fig. 1).

In the subsequent text, both indicial and nonindicial notations for the displacements will be used interchangeably without a preliminary notice, the correspondence between them being established as follows: $u_1 = u$, $u_2 = v$, and $u_3 = w$.

As energy-conjugate measures of strain and stress, we use the Green–Lagrange strain tensor and the second Piola–Kirchhoff stress tensor. We limit our research to the important case of small strains, moderate displacements (of the order of thickness of the plate), and moderate rotations (10–15 deg). This means that of all of the higher-order terms in the Green–Lagrange strain-displacement relations

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{s,i}u_{s,j}) \tag{1}$$

only $u_{3,\alpha}u_{3,\beta}$ (α, $\beta = 1, 2$) are not negligible compared to $u_{\alpha,i}$ ($\alpha = 1, 2$; $i = 1, 2, 3$).[11,13] Therefore, the strain–displacement relations become

$$\varepsilon_{xx} = u_{,x} + \tfrac{1}{2}(w_{,x})^2 \tag{2}$$

$$\varepsilon_{yy} = v_{,y} + \tfrac{1}{2}(w_{,y})^2 \tag{3}$$

$$\varepsilon_{xy} = \tfrac{1}{2}(u_{,y} + v_{,x} + w_{,x}w_{,y}) \tag{4}$$

$$\varepsilon_{xz} = \tfrac{1}{2}(u_{,z} + w_{,x}) \tag{5}$$

$$\varepsilon_{yz}^{(k)} = \tfrac{1}{2}(v_{,z} + w_{,y}) \tag{6}$$

$$\varepsilon_{zz} = w_{,z} \tag{7}$$

Now we need to find the simplified equations of motion and boundary conditions, such that their accuracy corresponds to the accuracy of the adopted von Kármán[13] strain–displacement relations. These equations of motion are used for computation of the transverse stresses in the postprocessing stage of the finite element analysis. We receive the equations of motion and boundary conditions, consistent with the von Kármán strain–displacement relations (2–7), using the virtual work principle (see Appendix). The equations of motion are written for each of the three conventional layers: the upper and lower face sheets and the core. The boundary conditions are applied to the upper and lower surfaces of the plate and to the interfaces between the face sheets and the core, with the result that at the lower surface,

$$\sigma_{xz}^{(1)} = 0, \qquad \sigma_{yz}^{(1)} = 0, \qquad \sigma_{zz}^{(1)} = -q_l \tag{8}$$

at the upper surface,

$$\sigma_{xz}^{(3)} = 0, \qquad \sigma_{yz}^{(3)} = 0, \qquad \sigma_{zz}^{(3)} = q_u \tag{9}$$

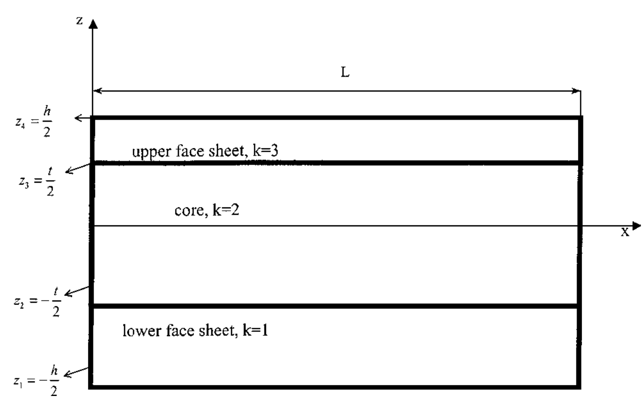

Fig. 1 Coordinate system and notations for the sandwich plates; axis *z* is in the thickness direction, *h* is a thickness of the whole plate, and *t* is a thickness of the core.

and at the interfaces,

$$\sigma_{xz}^{(1)}(z_2) = \sigma_{xz}^{(2)}(z_2), \qquad \sigma_{yz}^{(1)}(z_2) = \sigma_{yz}^{(2)}(z_2)$$
$$\sigma_{zz}^{(1)}(z_2) = \sigma_{zz}^{(2)}(z_2) \tag{10}$$

$$\sigma_{xz}^{(2)}(z_3) = \sigma_{xz}^{(3)}(z_3), \qquad \sigma_{yz}^{(2)}(z_3) = \sigma_{yz}^{(3)}(z_3)$$
$$\sigma_{zz}^{(2)}(z_3) = \sigma_{zz}^{(3)}(z_3) \tag{11}$$

In the laminate coordinate system (x, y, z), whose axes are aligned with the sides of the plate, the stress–strain relations for an orthotropic material have the form[11]

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & 0 & 0 & \bar{C}_{26} \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & 0 & 0 & \bar{C}_{36} \\ 0 & 0 & 0 & \bar{C}_{44} & \bar{C}_{45} & 0 \\ 0 & 0 & 0 & \bar{C}_{45} & \bar{C}_{55} & 0 \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{36} & 0 & 0 & \bar{C}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{Bmatrix} \tag{12}$$

or

$$\{\sigma\} = [\bar{C}]\{\varepsilon\} \tag{13}$$

where $\bar{C}_{ij}$ are the elastic coefficients, referred to the laminate coordinate system.

In addition, the displacements must be continuous at the interfaces between the faces and the core:

$$u_i^{(1)}\Big|_{z=z_2} = u_i^{(2)}\Big|_{z=z_2}, \qquad u_i^{(2)}\Big|_{z=z_3} = u_i^{(3)}\Big|_{z=z_3} \tag{14}$$

Two-Dimensional Plate Theory

Simplifying Assumptions of the Plate Theory

To construct a two-dimensional plate theory, we make simplifying assumptions regarding a distribution of the transverse strain components in the thickness direction. We assume that within the face sheets and the core the transverse strains do not depend on the z coordinate, but they can be different functions of coordinates x and y and time t in different face sheets and the core:

$$\varepsilon_{xz}^{(k)} = \varepsilon_{xz}^{(k)}(x, y, t), \qquad \varepsilon_{yz}^{(k)} = \varepsilon_{yz}^{(k)}(x, y, t)$$
$$\varepsilon_{zz}^{(k)} = \varepsilon_{zz}^{(k)}(x, y, t), \qquad (k = 1, 2, 3) \tag{15}$$

where the superscript k denotes the number of a sublaminate: $k = 1$ denotes the lower face sheet, $k = 2$ denotes the core, and $k = 3$ denotes the upper face sheet. Accuracy of the theory, based on these assumptions, is studied in the paper of Perel and Palazotto.[12]

The assumed transverse strains of Eqs. (15), together with displacements of the middle surface of the plate

$$u_0(x, y, t) \equiv u^{(2)}\Big|_{z=0}, \qquad v_0(x, y, t) \equiv v^{(2)}\Big|_{z=0}$$
$$w_0(x, y, t) \equiv w^{(2)}\Big|_{z=0} \tag{16}$$

are the unknown functions of the problem that will be computed by the finite element method. Therefore, all displacements, strains, and stresses must be expressed in terms of these functions.

Displacements in Terms of the Unknown Functions

To obtain expressions for the displacements in terms of the unknown functions $\varepsilon_{xz}^{(k)}$, $\varepsilon_{yz}^{(k)}$, $\varepsilon_{zz}^{(k)}$, u_0, v_0, and w_0, the strain–displacement relations (5–7) are integrated with the following result:

$$w^{(1)}(x, y, z, t) = w_0(x, y, t) + \varepsilon_{zz}^{(2)}(x, y, t)z_2 + \varepsilon_{zz}^{(1)}(x, y, t)(z - z_2) \qquad (z_1 \le z \le z_2) \tag{17}$$

$$w^{(2)}(x, y, z, t) = w_0(x, y, t) + \varepsilon_{zz}^{(2)}(x, y, t)z \tag{18}$$

$$w^{(3)}(x, y, z, t) = w_0(x, y, t) + \varepsilon_{zz}^{(2)}(x, y, t)z_3 + \varepsilon_{zz}^{(3)}(x, y, t)(z - z_3) \qquad (z_3 \le z \le z_4) \tag{19}$$

$$u^{(1)} = u_0 + \left(2\varepsilon_{xz}^{(2)} - w_{0,x}\right)z_2 - \tfrac{1}{2}\varepsilon_{zz,x}^{(2)}z_2^2 + \left(2\varepsilon_{xz}^{(1)} - w_{0,x} - \varepsilon_{zz,x}^{(2)}z_2\right)(z - z_2) - \tfrac{1}{2}\varepsilon_{zz,x}^{(1)}(z - z_2)^2 \qquad (z_1 \le z \le z_2) \tag{20}$$

$$u^{(2)} = u_0 + \left(2\varepsilon_{xz}^{(2)} - w_{0,x}\right)z - \tfrac{1}{2}\varepsilon_{zz,x}^{(2)}z^2 \qquad (z_2 \le z \le z_3) \tag{21}$$

$$u^{(3)} = u_0 + \left(2\varepsilon_{xz}^{(2)} - w_{0,x}\right)z_3 - \tfrac{1}{2}\varepsilon_{zz,x}^{(2)}z_3^2 + \left(2\varepsilon_{xz}^{(3)} - w_{0,x} - \varepsilon_{zz,x}^{(2)}z_3\right)(z - z_3) - \tfrac{1}{2}\varepsilon_{zz,x}^{(3)}(z - z_3)^2 \qquad (z_3 \le z \le z_4) \tag{22}$$

$$v^{(1)} = v_0 + \left(2\varepsilon_{yz}^{(2)} - w_{0,y}\right)z_2 - \tfrac{1}{2}\varepsilon_{zz,y}^{(2)}z_2^2 + \left(2\varepsilon_{yz}^{(1)} - w_{0,y} - \varepsilon_{zz,y}^{(2)}z_2\right)(z - z_2) - \tfrac{1}{2}\varepsilon_{zz,y}^{(1)}(z - z_2)^2 \qquad (z_1 \le z \le z_2) \tag{23}$$

$$v^{(2)} = v_0 + \left(2\varepsilon_{yz}^{(2)} - w_{0,y}\right)z - \tfrac{1}{2}\varepsilon_{zz,y}^{(2)}z^2 \qquad (z_2 \le z \le z_3) \tag{24}$$

$$v^{(3)} = v_0 + \left(2\varepsilon_{yz}^{(2)} - w_{0,y}\right)z_3 - \tfrac{1}{2}\varepsilon_{zz,y}^{(2)}z_3^2 + \left(2\varepsilon_{yz}^{(3)} - w_{0,y} - \varepsilon_{zz,y}^{(2)}z_3\right)(z - z_3) - \tfrac{1}{2}\varepsilon_{zz,y}^{(3)}(z - z_3)^2 \qquad (z_3 \le z \le z_4) \tag{25}$$

It can be verified easily that these expressions for the displacements are continuous across the boundaries between the faces and the core, that is, at $z = z_2$ and $z = z_3$.

Strains in Terms of the Unknown Functions

Expressions for the in-plane strains $\varepsilon_{xx}^{(k)}$, $\varepsilon_{xy}^{(k)}$, and $\varepsilon_{yy}^{(k)}$ in terms of the unknown functions are obtained by substituting expressions (17–25) for displacements in terms of the unknown functions into the strain–displacement relations (2), (3), and (4). The transverse strains $\varepsilon_{xz}^{(k)}$, $\varepsilon_{yz}^{(k)}$, and $\varepsilon_{zz}^{(k)}$ are the unknown functions themselves.

Extended Hamilton's Principle, Written for This Specific Problem

To derive either differential equations for the unknown functions with boundary conditions, or the finite element formulation, one can use the extended Hamilton's principle

$$\delta \int_{t_1}^{t_2} (T - \Pi)\, dt + \int_{t_1}^{t_2} \delta' W_{nc}\, dt = 0 \tag{26}$$

where T is a kinetic energy of the system, Π is a total potential energy of the system, and $\delta' W_{nc}$ is a virtual work of external nonconservative forces. Therefore, the extended Hamilton's principle for the sandwich plate on an elastic foundation can be written as follows:

$$\begin{aligned} \delta \int_{t_1}^{t_2} [&(\text{kinetic energy of plate}) - (\text{strain energy of plate}) \\ &- (\text{strain energy of elastic foundation}) \\ &- (\text{potential energy of plate in gravity field})]\, dt \\ &+ \int_{t_1}^{t_2} (\text{virtual work of damping forces})\, dt \\ &+ \int_{t_1}^{t_2} (\text{virtual work of surface forces})\, dt = 0 \end{aligned} \tag{27}$$

To derive the differential equations for the unknown functions with the boundary conditions, or to obtain a finite element formulation, all terms in the Hamilton's principle (27) need to be written in terms of the unknown functions u_0, v_0, w_0, $\varepsilon_{xz}^{(1)}$, $\varepsilon_{yz}^{(1)}$, $\varepsilon_{zz}^{(1)}$, $\varepsilon_{xz}^{(2)}$, $\varepsilon_{yz}^{(2)}$, $\varepsilon_{zz}^{(2)}$, $\varepsilon_{xz}^{(3)}$, $\varepsilon_{yz}^{(3)}$, and $\varepsilon_{zz}^{(3)}$.

Kinetic Energy of the Sandwich Plate

When it is considered that the mass density of the face sheets is constant, kinetic energy of the lower face sheet, core, and the upper face sheet can be written as follows:

$$T^{(k)} = \frac{1}{2}\rho^{(k)} \iiint_{(V^{(k)})} \begin{Bmatrix} \dot{u}^{(k)} \\ \dot{v}^{(k)} \\ \dot{w}^{(k)} \end{Bmatrix}^T \begin{Bmatrix} \dot{u}^{(k)} \\ \dot{v}^{(k)} \\ \dot{w}^{(k)} \end{Bmatrix} \mathrm{d}V \qquad (k = 1, 2, 3) \tag{28}$$

where dots over the letters denote partial derivatives with respect to time. The displacements in Eq. (28) are expressed in terms of the unknown functions by formulas (17–25). The kinetic energy of the sandwich plate is the sum of kinetic energies of the face sheets and the core:

$$T = T^{(1)} + T^{(2)} + T^{(3)} \tag{29}$$

Strain Energy of the Sandwich Plate

The face sheets of a sandwich plate are made from composite laminates, which are built up of fiber-reinforced plies. The orientation of the fibers can vary from ply to ply, and, therefore, values of the stiffness coefficients $\bar{C}_{ij}$ in the Hooke's law (referred to the laminate coordinate system) can vary from ply to ply in the face sheets. Let us introduce the following notation for a stiffness coefficient in the Hooke's law for a ply of the lower face sheet, in the laminate coordinate system:

$$^{\alpha}\bar{C}_{ij}^{(1)} \tag{30}$$

where the right superscript (1) denotes that a stiffness coefficient is associated with the first sublaminate, that is, the lower face sheet, the left superscript α is a number of a ply in a lower face sheet, and subscripts i and j denote a position of the stiffness coefficient in the stiffness matrix. The stiffness matrix with components $^{\alpha}\bar{C}_{ij}^{(1)}$ will be denoted as $[\bar{C}_{\alpha}^{(1)}]$. Thus, the strain energy of a lower face sheet's ply with a number α is

$$U_{\alpha}^{(1)} = \frac{1}{2} \iiint_{(V_{\alpha}^{(1)})} \{\varepsilon^{(1)}\}^T \left[\bar{C}_{\alpha}^{(1)}\right] \{\varepsilon^{(1)}\} \, \mathrm{d}V \tag{31}$$

where $V_{\alpha}^{(1)}$ is volume of a ply with number α, of the lower face sheet, and the column matrix of strains $\{\varepsilon^{(1)}\}$ is defined as follows:

$$\{\varepsilon^{(1)}\} \equiv \lfloor \varepsilon_{xx}^{(1)} \quad \varepsilon_{yy}^{(1)} \quad \varepsilon_{zz}^{(1)} \quad 2\varepsilon_{yz}^{(1)} \quad 2\varepsilon_{xz}^{(1)} \quad 2\varepsilon_{xy}^{(1)} \rfloor^T \tag{32}$$

Unlike the material coefficients $^{\alpha}\bar{C}_{ij}^{(1)}$, the strains do not have a subscript α, which denotes a number of a ply of the lower face sheet, because assumptions about through-the-thickness variation of strains are made for the whole lower face sheet, not for each individual ply of the lower face sheet. Therefore, each strain in the lower face sheet, as a function of the z coordinate, is represented with a single expression through the thickness of the lower face sheet.

The strain energy $U^{(1)}$ of the whole lower face sheet is a sum of strain energies of the plies of the lower face sheet:

$$U^{(1)} = \sum_{\alpha=1}^{n} U_{\alpha}^{(1)} \tag{33}$$

Similarly, one can write an expression for the strain energy of the upper face sheet $U^{(3)}$. The core of the sandwich plate is considered to be a homogeneous orthotropic medium. However, the failure in the core can be distributed nonuniformly in the thickness direction. As a result of this, in the presence of failure, the coefficients $\bar{C}_{ij}$ of the stress–strain relation of the core can vary in the thickness direction. To take account of this, the core is nominally divided into layers parallel to the x–y plane, such that within each layer the coefficients of the stress–strain relation can be considered approximately constant in the thickness direction. Thus, the core is treated as a laminated plate, the same way as the face sheets. The strain energy of the sandwich plate is the sum of the strain energies of the core and the face sheets

$$U = U^{(1)} + U^{(2)} + U^{(3)} \tag{34}$$

Potential Energy of the Sandwich Plate in the Gravity Field

The potential energy of the sandwich plate in the gravity field Π_g is equal to the sum of potential energies of the lower face $\Pi_g^{(1)}$, the core $\Pi_g^{(2)}$, and the upper face $\Pi_g^{(3)}$:

$$\Pi_g = \Pi_g^{(1)} + \Pi_g^{(2)} + \Pi_g^{(3)} \tag{35}$$

where

$$\Pi_g^{(k)} = \rho^{(k)} g \int_0^B \int_0^L \int_{z_k}^{z_{k+1}} w^{(k)} \, \mathrm{d}z \, \mathrm{d}x \, \mathrm{d}y \tag{36}$$

Strain Energy of Elastic Foundation

The strain energy of the elastic foundation, modeled as a Winkler foundation, is defined by the expression

$$U_f = \frac{1}{2} \int_0^B \int_0^L s(x) \left(w^{(1)} \Big|_{z=z_1} \right)^2 \mathrm{d}x \, \mathrm{d}y \tag{37}$$

where $s(x)$ is a modulus of the foundation.

Virtual Work of Surface Forces

It is assumed that the upper and lower surfaces of the plate are loaded by distributed forces in the transverse direction (along the z axis). Let $q_u\,(x, y, t)$ and $q_l\,(x, y, t)$ be forces per unit area in the transverse direction, acting on the plate's upper and lower surfaces, respectively. Then the virtual work $\delta' W$ of these forces is

$$\delta' W_s = \int_0^B \int_0^L q_u(x, y, t) \delta w^{(3)} \Big|_{z=z_4} \mathrm{d}x \, \mathrm{d}y + \int_0^B \int_0^L q_l(x, y, t) \delta w^{(1)} \Big|_{z=z_1} \mathrm{d}x \, \mathrm{d}y \tag{38}$$

Virtual Work of Damping Forces

The damping force per unit volume will be denoted as $\mathbf{\Phi}$. We will consider, because it is generally accepted, that the damping force is proportional to the velocity. Then, for the kth sublaminate, we can write the following expression for the column matrix of components of the damping force per unit volume:

$$\begin{Bmatrix} \Phi_x^{(k)} \\ \Phi_y^{(k)} \\ \Phi_z^{(k)} \end{Bmatrix} = -\beta^{(k)} \rho^{(k)} \frac{\partial}{\partial t} \begin{Bmatrix} u^{(k)} \\ v^{(k)} \\ w^{(k)} \end{Bmatrix} \tag{39}$$

where $\beta^{(k)}$ and $\rho^{(k)}$ are, respectively, a damping parameter and mass density of the kth sublaminate. The virtual work of the damping force in the kth sublaminate can be written as follows:

$$\delta' W_d^{(k)} = \int_0^B \int_0^L \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \delta u^{(k)} \\ \delta v^{(k)} \\ \delta w^{(k)} \end{Bmatrix}^T \begin{Bmatrix} \Phi_x^{(k)} \\ \Phi_y^{(k)} \\ \Phi_z^{(k)} \end{Bmatrix} \mathrm{d}z \, \mathrm{d}x \, \mathrm{d}y \tag{40}$$

The virtual work of the damping forces in the whole sandwich plate is the sum of the virtual works in the face sheets and the core:

$$\delta' W_d = \delta' W_d^{(1)} + \delta' W_d^{(2)} + \delta' W_d^{(3)} \tag{41}$$

The extended Hamilton's principle, written in terms of the unknown functions, can be used for deriving either differential equations and boundary conditions for the unknown functions, or it can be used for a finite element formulation. In the following section, we will develop a finite element formulation for the sandwich plate in cylindrical bending.

Finite Element Formulation for Cylindrical Bending

Let us call the dimension of the plate in the x direction the length and the dimension in the y direction the width. If the width of the plate is much larger than its length, and if the load intensity does not vary in the width direction, then the displacements do not depend on the y coordinate:

$$u = u(x, z), \qquad v = v(x, z), \qquad w = w(x, z) \tag{42}$$

Such a condition is called a generalized plane strain or cylindrical bending. In case of cylindrical bending, the unknown functions depend only on the x coordinate and time. Therefore, the two-dimensional plate-bending problem reduces to the one-dimensional problem. Then, all derivatives with respect to the y coordinate vanish in all formulas of the preceding sections.

Hereafter, we will consider a sandwich plate with an isotropic or transversely isotropic core and with face sheets being composite laminates of 0- and 90-deg plies orientation. In this case, at each point of the plate there is a plane of elastic symmetry parallel to the x–z coordinate plane, and, therefore, the condition of the generalized plane strain reduces to the condition of pure plane strain,[14] that is,

$$u = u(x, z), \qquad w = w(x, z), \qquad v = 0 \tag{43}$$

and all strain components, associated with the y direction, are equal to zero:

$$\varepsilon_{yy} = 0, \qquad \varepsilon_{xy} = 0, \qquad \varepsilon_{yz} = 0 \tag{44}$$

Therefore, the unknown functions are

$$\begin{array}{cccc} u_0, & w_0, & \varepsilon_{xz}^{(1)}, & \varepsilon_{zz}^{(1)} \\ \varepsilon_{xz}^{(2)}, & \varepsilon_{zz}^{(2)}, & \varepsilon_{xz}^{(3)}, & \varepsilon_{zz}^{(3)} \end{array} \tag{45}$$

To perform a finite element formulation, we represent the unknown functions $\varepsilon_{xz}^{(k)}(x, t)$, $\varepsilon_{zz}^{(k)}(x, t)$, $u_0(x, t)$, and $w_0(x, t)$ by piecewise interpolation polynomials:

$$u_0 = \lfloor M_1 \quad M_2 \rfloor \begin{Bmatrix} u(0) \\ u(l) \end{Bmatrix} \tag{46}$$

$$\varepsilon_{xz}^{(k)} = \lfloor M_1 \quad M_2 \rfloor \begin{Bmatrix} \varepsilon_{xz}^{(k)}(0) \\ \varepsilon_{xz}^{(k)}(l) \end{Bmatrix} \tag{47}$$

where

$$M_1 = 1 - x/l, \qquad M_2 = x/l \tag{48}$$

and l is a length of a finite element

$$w_0 = \lfloor N_1 \quad N_2 \quad N_3 \quad N_4 \rfloor \begin{Bmatrix} w_0(0) \\ \dfrac{\mathrm{d}w_0}{\mathrm{d}x}(0) \\ w_0(l) \\ \dfrac{\mathrm{d}w_0}{\mathrm{d}x}(l) \end{Bmatrix} \tag{49}$$

$$\varepsilon_{zz}^{(k)} = \lfloor N_1 \quad N_2 \quad N_3 \quad N_4 \rfloor \begin{Bmatrix} \varepsilon_{zz}^{(k)}(0) \\ \dfrac{\mathrm{d}\varepsilon_{zz}^{(k)}}{\mathrm{d}x}(0) \\ \varepsilon_{zz}^{(k)}(l) \\ \dfrac{\mathrm{d}\varepsilon_{zz}^{(k)}}{\mathrm{d}x}(l) \end{Bmatrix} \tag{50}$$

where

$$\begin{aligned} N_1 &= 1 - 3x^2/l^2 + 2x^3/l^3, \qquad & N_2 &= x - 2x^2/l + x^3/l^2 \\ N_3 &= 3x^2/l^2 - 2x^3/l^3, \qquad & N_4 &= -x^2/l + x^3/l^2 \end{aligned} \tag{51}$$

Here and further in this section, devoted to the finite element formulation, it is implied for simplicity of notations that x is a coordinate in the element (local) coordinate system, the origin of which coincides with a left node of a finite element.

Thus, the combined finite element has 24 degrees of freedom. At each node there are 12 nodal parameters: u_0, $\varepsilon_{xz}^{(1)}$, $\varepsilon_{xz}^{(2)}$, $\varepsilon_{xz}^{(3)}$, w_0, $\mathrm{d}w_0/\mathrm{d}x$, $\varepsilon_{zz}^{(1)}$, $\mathrm{d}\varepsilon_{zz}^{(1)}/\mathrm{d}x$, $\varepsilon_{zz}^{(2)}$, $\mathrm{d}\varepsilon_{zz}^{(2)}/\mathrm{d}x$, $\varepsilon_{zz}^{(3)}$, and $\mathrm{d}\varepsilon_{zz}^{(3)}/\mathrm{d}x$.

From the extended Hamilton's principle (27), written in terms of nodal parameters d_i, $i = 1, 2, \ldots, 24$, the following equation of motion of a finite element in terms of the nodal parameters can be derived:

$$\underset{(24\times24)}{[m]}\ \underset{(24\times1)}{\{\ddot{d}\}} + \underset{(24\times24)}{[c]}\ \underset{(24\times1)}{\{\dot{d}\}} + \underset{(24\times24)}{[k]}\ \underset{(24\times1)}{\{d\}} + \underset{(24\times1)}{\frac{\partial U_{\mathrm{nl}}}{\partial\{d\}}} = \underset{(24\times1)}{\{r\}} \tag{52}$$

In this equation, a part U_{nl} of the strain energy is due to the nonlinear terms in the strain-displacement relations (geometric nonlinearity of the von Kármán[13] type). The expression for U_{nl} is not a quadratic form of the nodal variables; therefore, the vector $\partial U_{\mathrm{nl}}/\partial\{d\}$ is not linear with respect to the nodal variables. The matrices that enter into Eq. (52) were derived with the use of exact integration, performed with a program for symbolic computation, MAPLE. In our finite element analysis, the global damping matrix is not assembled from the element damping matrices. Instead, the proportional damping model is used, in which the global damping matrix $[C]$ is presented as a linear combination of the global stiffness and mass matrices. [Therefore, the virtual work of the damping forces, as presented by Eqs. (35–37), is not used in our finite element formulation, but has only a theoretical importance in formulating the two-dimensional plate-type theory.] Thus,

$$[C] = \alpha_1[K] + \alpha_2[M] \tag{53}$$

where α and β are constants to be determined from two given logarithmic decrements of damping, δ_1 and δ_2, which correspond to two unequal frequencies of vibrations ω_1 and ω_2 by the formulas

$$\alpha_1 = \frac{\delta_1\omega_1 - \delta_2\omega_2}{\pi\left(\omega_1^2 - \omega_2^2\right)} \tag{54}$$

$$\alpha_2 = \frac{\omega_1\omega_2(\delta_2\omega_1 - \delta_1\omega_2)}{\pi\left(\omega_1^2 - \omega_2^2\right)} \tag{55}$$

The assembled equations of motion of the whole structure were solved by direct integration with the use of the Newmark method (see Ref. 11).

In the postprocessing stage of the finite element analysis, the in-plane stresses $\sigma_{xx}^{(k)}$, $\sigma_{xy}^{(k)}$, and $\sigma_{yy}^{(k)}$ are computed with the use of the constitutive equations, which, in the case of pure plane strain (cylindrical bending), take the form

$$\sigma_{xx}^{(k)} = \bar{C}_{11}^{(k)}\varepsilon_{xx}^{(k)} + \bar{C}_{13}^{(k)}\varepsilon_{zz}^{(k)} \tag{56}$$

$$\sigma_{xy}^{(k)} = \bar{C}_{16}^{(k)}\varepsilon_{xx}^{(k)} + \bar{C}_{36}^{(k)}\varepsilon_{zz}^{(k)} \tag{57}$$

$$\sigma_{yy}^{(k)} = \bar{C}_{12}^{(k)}\varepsilon_{xx}^{(k)} + \bar{C}_{23}^{(k)}\varepsilon_{zz}^{(k)} \tag{58}$$

whereas the transverse stresses $\sigma_{xz}^{(k)}$, $\sigma_{yz}^{(k)}$, and $\sigma_{zz}^{(k)}$ need to be computed by integration of the pointwise equations of motion, which, in the case of pure plane strain and under the assumption that the plate is perpendicular to the direction of gravity force, take the form

$$\sigma_{xx,x}^{(k)} + \sigma_{xz,z}^{(k)} = \rho^{(k)}\ddot{u}^{(k)} \tag{59}$$

$$\sigma_{yx,x}^{(k)} + \sigma_{yz,z}^{(k)} = 0 \tag{60}$$

$$\sigma_{zx,x}^{(k)} + \sigma_{zz,z}^{(k)} + \frac{\partial}{\partial x}\left(\sigma_{xx}^{(k)}w_{,x}^{(k)}\right) - \rho^{(k)}g = \rho^{(k)}\ddot{w}^{(k)}, \qquad (k = 1, 2, 3) \tag{61}$$

If one integrates Eqs. (59–61) and satisfies stress boundary conditions on the lower surfaces [Eqs. (8)] and conditions of continuity of the transverse stresses [Eqs. (10) and (11)], one gets

$$\sigma_{xz}^{(1)} = \int_{z_1}^{z} \left(\rho^{(1)}\ddot{u}^{(1)} - {}^{H}\sigma_{xx,x}^{(1)}\right) \mathrm{d}z \qquad (z_1 \le z \le z_2) \tag{62}$$

$$\sigma_{xz}^{(2)} = \int_{z_1}^{z_2} \left(\rho^{(1)}\ddot{u}^{(1)} - {}^{H}\sigma_{xx,x}^{(1)}\right) \mathrm{d}z + \int_{z_2}^{z} \left(\rho^{(2)}\ddot{u}^{(2)} - {}^{H}\sigma_{xx,x}^{(2)}\right) \mathrm{d}z \qquad (z_2 \le z \le z_3) \tag{63}$$

$$\sigma_{xz}^{(3)} = \int_{z_1}^{z_2} \left(\rho^{(1)}\ddot{u}^{(1)} - {}^{H}\sigma_{xx,x}^{(1)}\right) \mathrm{d}z + \int_{z_2}^{z_3} \left(\rho^{(2)}\ddot{u}^{(2)} - {}^{H}\sigma_{xx,x}^{(2)}\right) \mathrm{d}z + \int_{z_3}^{z} \left(\rho^{(3)}\ddot{u}^{(3)} - {}^{H}\sigma_{xx,x}^{(3)}\right) \mathrm{d}z \qquad (z_3 \le z \le z_4) \tag{64}$$

$$\sigma_{yz}^{(1)}(x,z,t) = -\int_{z_1}^{z} {}^{H}\sigma_{yx,x}^{(1)} \,\mathrm{d}z \tag{65}$$

$$\sigma_{yz}^{(2)}(x,z,t) = -\int_{z_1}^{z_2} {}^{H}\sigma_{yx,x}^{(1)} \,\mathrm{d}z - \int_{z_2}^{z} {}^{H}\sigma_{yx,x}^{(2)} \,\mathrm{d}z \tag{66}$$

$$\sigma_{yz}^{(3)}(x,z,t) = -\int_{z_1}^{z_2} {}^{H}\sigma_{yx,x}^{(1)} \,\mathrm{d}z - \int_{z_2}^{z_3} {}^{H}\sigma_{yx,x}^{(2)} \,\mathrm{d}z - \int_{z_3}^{z} {}^{H}\sigma_{yx,x}^{(3)} \,\mathrm{d}z \tag{67}$$

$$\sigma_{zz}^{(1)} = -\frac{q_l}{b} + \int_{z_1}^{z} \left[\rho^{(1)}\left(\ddot{w}^{(1)} + g\right) - \frac{\partial}{\partial x}\left({}^{H}\sigma_{xx}^{(1)} w_{,x}^{(1)}\right) - \sigma_{zx,x}^{(1)}\right] \mathrm{d}z \tag{68}$$

$$\sigma_{zz}^{(2)} = -\frac{q_l}{b} + \int_{z_1}^{z_2} \left[\rho^{(1)}\left(\ddot{w}^{(1)} + g\right) - \frac{\partial}{\partial x}\left({}^{H}\sigma_{xx}^{(1)} w_{,x}^{(1)}\right) - \sigma_{zx,x}^{(1)}\right] \mathrm{d}z + \int_{z_2}^{z} \left[\rho^{(2)}\left(\ddot{w}^{(2)} + g\right) - \frac{\partial}{\partial x}\left({}^{H}\sigma_{xx}^{(2)} w_{,x}^{(2)}\right) - \sigma_{zx,x}^{(2)}\right] \mathrm{d}z \tag{69}$$

$$\sigma_{zz}^{(3)} = -\frac{q_l}{b} + \int_{z_1}^{z_2} \left[\rho^{(1)}\left(\ddot{w}^{(1)} + g\right) - \frac{\partial}{\partial x}\left({}^{H}\sigma_{xx}^{(1)} w_{,x}^{(1)}\right) - \sigma_{zx,x}^{(1)}\right] \mathrm{d}z + \int_{z_2}^{z_3} \left[\rho^{(2)}\left(\ddot{w}^{(2)} + g\right) - \frac{\partial}{\partial x}\left({}^{H}\sigma_{xx}^{(2)} w_{,x}^{(2)}\right) - \sigma_{zx,x}^{(2)}\right] \mathrm{d}z + \int_{z_3}^{z} \left[\rho^{(3)}\left(\ddot{w}^{(3)} + g\right) - \frac{\partial}{\partial x}\left({}^{H}\sigma_{xx}^{(3)} w_{,x}^{(3)}\right) - \sigma_{zx,x}^{(3)}\right] \mathrm{d}z \tag{70}$$

In Eqs. (62–70), the in-plane stresses with the left superscripts H (which stands for Hooke's law) are computed with the use of the Hooke's law, that is, from equations (56–58). As can be seen from Eqs. (62–70), the inertia and nonlinear terms are taken into account in the expressions for the transverse stresses.

With the use of Eqs. (56–58) and (62–70), all stress components can be expressed in terms of the unknown functions u_0, w_0, $\varepsilon_{xz}^{(1)}$, $\varepsilon_{zz}^{(1)}$, $\varepsilon_{xz}^{(2)}$, $\varepsilon_{zz}^{(2)}$, $\varepsilon_{xz}^{(3)}$, and $\varepsilon_{zz}^{(3)}$. The expressions for the stresses in terms of the unknown functions are not shown explicitly because of their large size. Examples of such expressions are shown in the paper of Perel and Palazotto.[12]

The values of u_0, w_0, $\partial w_0/\partial x$, $\varepsilon_{zz}^{(k)}$, $\partial\varepsilon_{zz}^{(k)}/\partial x$, and $\varepsilon_{xz}^{(k)}$ are most accurate at the nodes of the finite element mesh (because these variables are carried as nodal variables), and for computation of stresses, they can be taken directly from the finite element solution. The second derivatives $\partial^2 w_0/\partial x^2$ and $\partial^2\varepsilon_{zz}^{(k)}/\partial x^2$, computed from interpolation polynomials, used in the finite element formulation, are most accurate at the Gauss points. (The locations of the minimal-error points of the derivatives of the field variables within a finite element were calculated with the use of a method presented by Akin.[15]) The third derivatives $\partial^3 w_0/\partial x^3$ and $\partial^3\varepsilon_{zz}^{(k)}/\partial x^3$, computed from the interpolation polynomials, are constant over the element's length and are most accurate in the middle of the finite element. The derivatives $\partial^4 w_0/\partial x^4$, $\partial^4\varepsilon_{zz}^{(2)}/\partial x^4$, $\partial^2 u_0/\partial x^2$, and $\partial^2\varepsilon_{xz}^{(k)}/\partial x^2$, taken as derivatives of the interpolation polynomials, which are used in the finite element formulation, are equal to zero, which can be wrong for a particular problem. Therefore, these derivatives are computed numerically at the nodal points by a finite difference scheme, using the nodal values of w_0, $\varepsilon_{zz}^{(k)}$, u_0, and $\varepsilon_{xz}^{(k)}$, obtained from the finite element solution. Thus, there are no points within the finite element where all of the derivatives of the field variables are most accurate simultaneously. Therefore, to compute stresses, the average (over the element's length) values of the field variables and their derivatives are evaluated, because average values at the points where these quantities are most accurate, and then substituted into the expressions for the stresses, producing average (over the element's length) values of stresses.

The computation of the transverse stresses from the equations of motion allows one to satisfy the stress boundary conditions on the upper and lower surfaces of the plate and the conditions of continuity of the transverse stresses at the interfaces between the faces and the core of the sandwich plate and between the plies with different material properties within the faces.[12] The computation of the transverse stresses by integration of the pointwise equilibrium equations is demonstrated, for example, in the work by Reddy[11] and some other works included in the bibliography therein. That the computation of the transverse stresses by integration of the pointwise equilibrium equations (or equations of motion) allows for the satisfaction of the stress boundary conditions not only on one of the external surfaces (upper or lower), but on both of them, was mentioned in the work of Perel and Palazotto.[12]

In most common cases of boundary conditions, that is, simply supported, clamped, and free edges of the plate, the nodal values of the transverse strains $\varepsilon_{xz}^{(1)}$, $\varepsilon_{zz}^{(1)}$, $\varepsilon_{xz}^{(2)}$, $\varepsilon_{zz}^{(2)}$, $\varepsilon_{xz}^{(3)}$, and $\varepsilon_{zz}^{(3)}$ and their derivatives need not be specified at the edges.

Time Integration with Account of Damage Progression

When a failure occurs in a single layer of a composite laminate, a composite structure can still carry a load. Therefore, a subsequent failure prediction is required to determine a dynamic response of the structure in the presence of some damage. This problem is dealt with by assuming that, within a finite element where the damage occurs, the original material characteristics of the damaged ply can be replaced with degraded material characteristics. The degraded material properties are assumed to be small fractions of the properties of the undamaged material, but not equal to zero, to avoid ill conditioning of the finite element equations. For example, a degraded value of the Young's modulus E_d of the damaged ply within a finite element is computed as

$$E_{1d} = (\mathrm{SRC})\mathrm{E}_1 \tag{71}$$

where E_1 is an original value of the Young's modulus, in this case multiplied by a stiffness reduction coefficient (SRC).

The face sheets of the sandwich plate are made of laminated composite plates that can fail in different modes due to matrix cracking, fiber fracture, fiber matrix debonds, and delamination. Therefore, for accurate prediction of failure in the face sheets, one needs to use a failure criterion that takes account of the microstructure of the composite laminates and the variety of modes of failure that can occur due to this microstructure. A set of failure criteria, designed for this purpose, were suggested by Hashin.[16] Therefore, for the face sheets, Hashin's criteria are used in this study.

The core of the sandwich plate, made of polymeric foam or a honeycomb structure, is modeled as a homogeneous isotropic or transversely isotropic medium. Such a medium has fewer modes of failure, namely, crushing under compression and cracking under tension. Therefore, for the failure analysis of the core, it is more appropriate to use a failure criterion that does not take account of the microstructure of the material. One such criterion is the Tsai–Wu criterion (see Refs. 17 and 18), and it is used for the core in our study. The core, which is uniform before the beginning of the damage, becomes nonuniform in the thickness direction (as well as in longitudinal direction) when the damage starts to progress in the thickness direction. For this reason, the core is divided into the nominal layers, and a check of the failure criterion in the middle of thickness of each such layer is carried out.

In the case of crushing of the core of the sandwich plate and tension modes of failure, the SRC for all material constants are set to be as small as possible, but their smallness is limited by the need to avoid numerical difficulties that can be caused by the large difference of values of material constants of adjacent finite elements. Such values of the stiffness reduction coefficients are found by numerical experimentation. In the numerical examples presented hereafter, the SRC is set equal to 0.001 in the case of crushing of the core of the sandwich plate and failure in tension and 0.01 in the case of fiber failure in compression in the face sheets. The stiffness reduction coefficient, associated with the fiber failure in compression, is set equal to a larger value because the compressive fiber mode of failure is interpreted as buckling of fibers in the matrix.[19] It is assumed that if the buckling of the fibers occurs, the layer still has some residual strength.

At each time step, the average (over a finite element length) stresses in each layer within each element are used in the failure criteria.

Now, the algorithm of taking account of damage progression will be presented without details of how it is embedded into the time integration scheme: (The details will be presented subsequently.)

1) At each time step of time integration, compute average (over an element length) stresses σ_{xx}, σ_{xy}, σ_{yy}, σ_{xz}, σ_{yz}, and σ_{zz} in the problem coordinate system in all finite elements, in the middle of each ply of the face sheets [at $z = (\xi_k + \xi_{k+1})/2$] and in the middle of each nominal layer of the core. (Computation of average stresses was discussed earlier.)

2) Transform the stresses to the principle material coordinates, that is, compute σ_{11}, σ_{22}, σ_{33}, σ_{12}, σ_{13}, and σ_{23}.

3) Substitute the stresses in the material coordinate system into the failure criteria. The Hashin[16] criteria are used for the face sheets and the Tsai–Wu criterion (Refs. 17 and 18) is used for the core. If the failure occurs, reduce the appropriate engineering constants of the face sheets and the core using the methods described earlier.

4) By the use of the modified values of engineering elastic constants, for each layer of each finite element that fails, recompute elastic constants ${}^{\alpha}\bar{C}_{ij}^{(k)}$, element stiffness matrices, the global stiffness matrix, and restart the analysis at the same time step, that is, return to step 1. Such a method is used because when failure occurs the stress field changes instantly due to the change of material properties. This redistribution of the stresses may cause additional failure to occur. Therefore, in case of failure, the time incrementation must be stopped, and analysis must be run again for the same time interval to determine the new failure. If the new failure does not occur, the analysis can go on to the next time step.

5) If failure does not occur, proceed to the next time step.

Analysis goes on for a number of time steps, specified by a user.

Now, the details will be presented on how the damage progression algorithm is embedded into the time integration scheme of the system equations of motion

$$[M]\{\ddot{\Theta}\} + [C]\{\dot{\Theta}\} + [K]\{\Theta\} + \{Q\} = \{P\} \tag{72}$$

with the use of the Newmark[20] method. In Eq. (72), the matrix $[K]$ is the system stiffness matrix, whose components do not depend on the nodal unknowns Θ_i, and $\{Q\}$ is a nonlinear part of the internal force vector, whose components are defined as $\partial(U_{\mathrm{nl}})_{\mathrm{system}}/\partial\Theta_i$, where $(U_{\mathrm{nl}})_{\mathrm{system}}$ is the whole system's part of the strain energy that is not quadratic with respect to the nodal unknowns. The part of the strain energy $(U_{\mathrm{nl}})_{\mathrm{system}}$ appears due to the nonlinear terms in the von Kármán[13] strain-displacement relations. Thus, the problem being solved numerically is geometrically nonlinear.

Let us introduce the following notations:

$$\{\Theta\}|_{t=t_n} \equiv \{\Theta\}_n$$

is the vector of nodal variables, evaluated at moment of time t_n, and

$$\{\Theta\}|_{t=t_{n+1}} \equiv \{\Theta\}_{n+1}$$

is the vector of nodal variables, evaluated at moment of time t_{n+1}, $\tau \equiv t_{n+1} - t_n$.

With the use of the Taylor series expansion, vectors $\{\Theta\}_{n+1}$ and $\{\dot{\Theta}\}_{n+1}$ can be written in the form

$$\{\Theta\}_{n+1} \approx \{\Theta\}_n + \tau\{\dot{\Theta}\}_n + \tau^2\left(\tfrac{1}{2} - \beta\right)\{\ddot{\Theta}\}_n + \tau^2\beta\{\ddot{\Theta}\}_{n+1} \tag{73}$$

$$\{\dot{\Theta}\}_{n+1} \approx \{\dot{\Theta}\}_n + \tau(1 - \gamma\tau)\{\ddot{\Theta}\}_n + \tau^2\gamma\{\ddot{\Theta}\}_{n+1} \tag{74}$$

where β and γ are free parameters that control the accuracy and stability of the method. In the example problems considered later, the values of these parameters were chosen to be $\beta = \frac{1}{4}$ and $\gamma = \frac{1}{2}$, which correspond to the method of constant mean acceleration. Such a method is unconditionally stable and provides a satisfactory accuracy.

Equations of motion (72), evaluated at a moment of time t_{n+1}, are

$$[M]\{\ddot{\Theta}\}_{n+1} + [C]\{\dot{\Theta}\}_{n+1} + [K]\{\Theta\}_{n+1} + \{Q\}_{n+1} = \{P\} \tag{75}$$

In Eqs. (72) and (75), the load vector $\{P\}$ is due to the gravity force, and so it does not depend on time and, therefore, does not have the subscript n. Substitution of Eqs. (73) and (74) into Eq. (75) and simple transformations yields

$$[\hat{K}]\{\Theta\}_{n+1} + \{Q\}_{n+1} = \{\hat{F}\}_n \tag{76}$$

where

$$[\hat{K}] \equiv [K] + (1/\tau^2\beta)[M] + (\gamma/\beta)[C] \tag{77}$$

$$\{\hat{F}\}_n = \{P\} - [C][\{\dot{\Theta}\}_n + \tau(1 - \gamma\tau)\{\ddot{\Theta}\}_n] + \{(1/\tau^2\beta)[M] + (\gamma/\beta)[C]\}\left[\{\Theta\}_n + \tau\{\dot{\Theta}\}_n + \tau^2\left(\tfrac{1}{2} - \beta\right)\{\ddot{\Theta}\}_n\right] \tag{78}$$

Now, assuming that we know the values of $\{\Theta\}_n$, $\{\dot{\Theta}\}_n$, and $\{\ddot{\Theta}\}_n$, we need to find the values of $\{\Theta\}_{n+1}$, $\{\dot{\Theta}\}_{n+1}$, and $\{\ddot{\Theta}\}_{n+1}$. Components of vector $\{Q\}_{n+1}$, which enter into Eq. (76), depend nonlinearly on components of the vector of nodal parameters $\{\Theta\}_{n+1}$. Therefore, Eq. (76) is a nonlinear system of algebraic equations with respect to components of the vector $\{\Theta\}_{n+1}$. These nonlinear equations are solved by a direct iteration (Picard) method (see Ref. 11).

The direct iteration method is based on computing a sequence of vectors

$$\{\Theta\}_{n+1}^{(1)}, \qquad \{\Theta\}_{n+1}^{(2)}, \qquad \{\Theta\}_{n+1}^{(3)}, \ldots \tag{79}$$

by solving a system of linear algebraic equations

$$[\hat{K}]\{\Theta\}_{n+1}^{(r+1)} = \{\hat{F}\}_n - \{Q\}_{n+1}^{(r)} \tag{80}$$

where the vector $\{Q\}_{n+1}^{(r)}$ is the vector $\{Q\}_{n+1}$ evaluated at $\{\Theta\}_{n+1} = \{\Theta\}_{n+1}^{(r)}$, that is, evaluated with the use of values of nodal parameters Θ_i obtained at the rth iteration. The components of the matrix $[\hat{K}]$ and the vector $\{\hat{F}\}_n$ do not depend on the unknowns, that is, on the components of the vector $\{\Theta\}_{n+1}$. If the sequence of vectors $\{\Theta\}_{n+1}^{(1)}$, $\{\Theta\}_{n+1}^{(2)}$, $\{\Theta\}_{n+1}^{(3)}, \ldots,$ converges to some vector $\{\tilde{\Theta}\}_{n+1}$, then this vector $\{\tilde{\Theta}\}_{n+1}$ is a solution of the system of algebraic equations (76). In our numerical implementation, the first term of the iteration sequence $\{\Theta\}_{n+1}^{(1)}$, $\{\Theta\}_{n+1}^{(2)}$, $\{\Theta\}_{n+1}^{(3)}, \ldots,$ is set equal to a zero vector at all time intervals:

$$\{\Theta\}_{n+1}^{(1)} = \{0\} \tag{81}$$

for $n = 1, 2, 3, \ldots.$ Iteration is stopped if a norm of vector $\{\Theta\}_{n+1}^{(r+1)} - \{\Theta\}_{n+1}^{(r)}$ (a difference of solution vectors in two successive approximations), divided by the norm of vector $\{\Theta\}_{n+1}^{(r+1)}$, is less than some number (tolerance):

$$\frac{\left\|\{\Theta\}_{n+1}^{(r+1)} - \{\Theta\}_{n+1}^{(r)}\right\|}{\left\|\{\Theta\}_{n+1}^{(r+1)}\right\|} < \text{tolerance} \tag{82}$$

As a norm of a vector, we used a square root of the sum of the squares of its components. Let $(\Theta_i)_{n+1}^{(r)}$ be an ith component of the approximate solution vector obtained in an iteration with a number

r at a moment of time with a number $n+1$. Then the criterion (82) for stopping the iterations will be written as follows:

$$\frac{\sqrt{\sum_i \left[(\Theta_i)_{n+1}^{(r+1)} - (\Theta_i)_{n+1}^{(r)}\right]^2}}{\sqrt{\sum_i \left[(\Theta_i)_{n+1}^{(r+1)}\right]^2}} < \text{tolerance} \qquad (83)$$

In the example problems considered hereafter, we set tolerance$=1\times 10^{-4}$.

Thus, in the problems with the damage progression taken into account, the algorithm of the Newmark[20] time integration scheme, combined with the direct iteration method of solving the nonlinear algebraic equations (76), can be summarized as follows:

1) At the first time interval $[t_1, t_2]$, set the vectors of initial generalized displacements $\{\Theta\}_1$ and velocities $\{\dot{\Theta}\}_1$ equal to the values specified in initial conditions. The vector $\{\ddot{\Theta}\}_1$ of initial generalized accelerations is found from the Eq. (75), where n is set equal to zero:

$$[M]\{\ddot{\Theta}\}_1 + [C]\{\dot{\Theta}\}_1 + [K]\{\Theta\}_1 + \{Q\}_1 = \{P\} \qquad (84)$$

This is a system of linear algebraic equations with respect to components of the vector $\{\ddot{\Theta}\}_1$.

2) At the nth time interval $[t_n, t_{n+1}]$, the vectors $\{\Theta\}_n$, $\{\dot{\Theta}\}_n$, $\{\ddot{\Theta}\}_n$, and $\{Q\}_n$ are known, and it is necessary to find the vectors $\{\Theta\}_{n+1}$, $\{\dot{\Theta}\}_{n+1}$, $\{\ddot{\Theta}\}_{n+1}$, and $\{Q\}_{n+1}$. For this purpose, the following algorithm is used:

a) Set the iteration counter r equal to 1, and set the initial approximation for the vector $\{\Theta\}_{n+1}$ to be a zero vector:

$$\{\Theta\}_{n+1}^{(1)} = \{0\} \qquad (85)$$

b) Evaluate $\{Q\}_{n+1}^{(r)}$, that is, evaluate $\{Q\}_{n+1}$ at $\{\Theta\}_{n+1} = \{\Theta\}_{n+1}^{(r)}$ and solve a linear system of algebraic equations for the components of the vector $\{\Theta\}_{n+1}^{(r+1)}$:

$$[\hat{K}]\{\Theta\}_{n+1}^{(r+1)} = \{\hat{F}\}_n - \{Q\}_{n+1}^{(r)} \qquad (86)$$

Evaluate the acceleration vector of the current iteration (iteration with number $r+1$) by the formula

$$\{\ddot{\Theta}\}_{n+1}^{(r+1)} = \frac{1}{\tau^2\beta}\left[\{\Theta\}_{n+1}^{(r+1)} - \{\Theta\}_n - \tau\{\dot{\Theta}\}_n - \tau^2\left(\tfrac{1}{2}-\beta\right)\{\ddot{\Theta}\}_n\right] \qquad (87)$$

[Equation (87) is obtained by expressing $\{\ddot{\Theta}\}_{n+1}$ from Eq. (73).] Evaluate the velocity vector of the current iteration (iteration with number $r+1$) by the formula

$$\{\dot{\Theta}\}_{n+1}^{(r+1)} = \{\dot{\Theta}\}_n + \tau(1-\gamma\tau)\{\ddot{\Theta}\}_n + \tau^2\gamma\{\ddot{\Theta}\}_{n+1}^{(r+1)} \qquad (88)$$

[Equation (88) is obtained from Eq. (74).]

c) Check if the vectors $\{\Theta\}_{n+1}^{(r+1)}$ and $\{\Theta\}_{n+1}^{(r)}$ satisfy the convergence criterion of equation (83).

If the convergence criterion is not satisfied, then begin a new iteration within this time interval, that is, set $r = r+1$ and go back to the step b. If the convergence criterion is satisfied, go to the next step.

d) Set the vector of nodal parameters and the vectors of the first and second time derivatives of the nodal parameters equal to the corresponding vectors obtained in the iteration at which the convergence criterion of the step c was satisfied, that is, set

$$\{\Theta\}_{n+1} = \{\Theta\}_{n+1}^{(r+1)} \qquad (89)$$

$$\{\dot{\Theta}\}_{n+1} = \{\dot{\Theta}\}_{n+1}^{(r+1)} \qquad (90)$$

$$\{\ddot{\Theta}\}_{n+1} = \{\ddot{\Theta}\}_{n+1}^{(r+1)} \qquad (91)$$

for use in the next time step and for computation of stresses at $t = t_{n+1}$.

e) Compute average stresses in all plies of each finite element at $t = t_{n+1}$, using the vectors $\{\Theta\}_{n+1}$, $\{\dot{\Theta}\}_{n+1}$, and $\{\ddot{\Theta}\}_{n+1}$, obtained in step d. Substitute these stresses into the failure criteria. If the failure occurs in a ply of a finite element, modify material elastic constants of this ply, modify the element stiffness matrix $[k]$ and the nonlinear internal force vector $\{q\}_{n+1} = (\partial U_{nl}/\partial\{\theta\})_{n+1}$ of the finite element to which the damaged ply belongs, and assemble the global stiffness matrix $[K]$ and global nonlinear internal force vector $\{Q\}_{n+1}$ with account of modifications to the element stiffness matrices and element nonlinear internal force vectors due to the damage. Then go to the step b, that is, recompute vectors $\{\Theta\}_{n+1}$, $\{\dot{\Theta}\}_{n+1}$, and $\{\ddot{\Theta}\}_{n+1}$ and stresses at the same moment of time.

(When failure occurs, the stress field changes instantly due to the change of material properties. This redistribution of the stresses may cause additional failure to occur. Therefore, in case of failure, the time incrementation must be stopped and analysis must be run again for the same time interval to determine the new failure. If the new failure does not occur, the analysis can go on to the next time step.)

If failure does not occur, set $n = n+1$, that is, go to the next time interval.

Analysis goes on for a number of time steps, specified by a user.

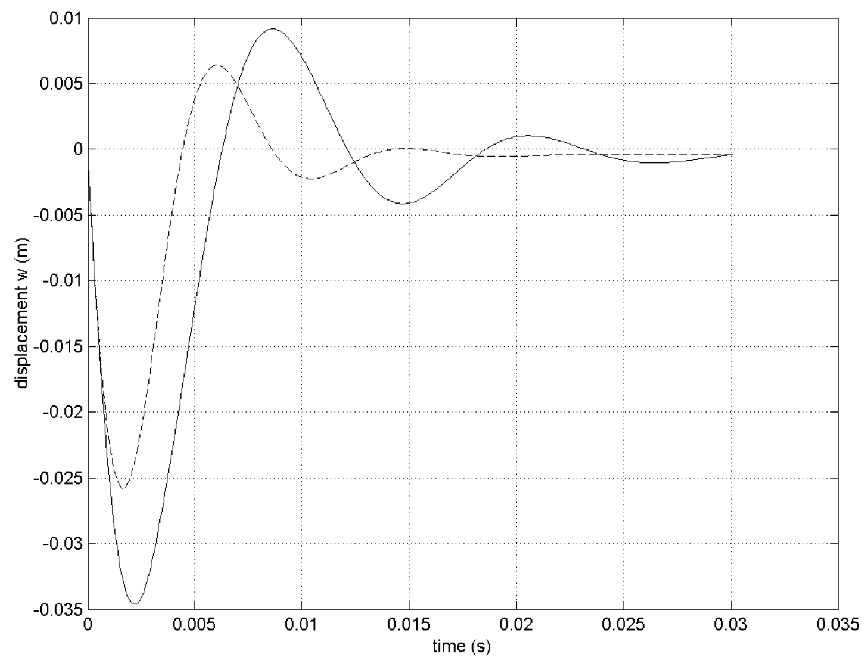

Fig. 2 Transverse displacement w, at $x = L/2$, $z = -h/2$, as a function of time, in a sandwich plate dropped on elastic foundation with initial velocity -30 m/s. The foundation modulus is 6.7864×10^8 Pa/m (clay); - - -, results of analysis without account of damage; and ——, with damage included.

Example Problems

In some problems, when the plates are loaded on both the upper and lower surfaces, or when the plates are on elastic foundations, transverse compressibility of the sandwich plates can not be neglected. In the proposed finite element formulation, the transverse compressibility is taken into account by assuming that the direct transverse strain ε_{zz} is not equal to zero and by including this strain into the expression for the strain energy. In the following example, we consider a sandwich plate, with a rigid body on its upper surface, under its impact against an elastic Winkler foundation, and demonstrate that the change of the plate's height during this impact can be captured by the finite element model. In the finite element formulation, the presence of the rigid body on the upper surface is taken into account by including a kinetic energy of the rigid body in Hamilton's principle. Example problems, considered hereafter are solved with the use of the geometrically nonlinear formulation. There are 40 finite elements used in all example problems.

Thus far, the numerical implementation of the theory is performed for the case of cylindrical bending only, which occurs if the width of the plate is larger than the length and if the load on the surface is uniformly distributed along the width. In this case, the stress distribution and stiffness degradation are uniform in the width direction.

Let us consider an example of a sandwich plate with laminated composite face sheets, made of AS4/3501-6 material, and a honeycomb core, made of Nomex HRH10-1/8-4.0. The material properties of the face sheets and the core, used in the example problems, are listed hereafter.

The elastic constants of the face sheets are as follows: $E_1 = 144.8\times10^9$ N/m^2, $E_2 = 9.7\times10^9$ N/m^2, $E_3 = 9.7\times10^9$ N/m^2, $G_{23} = 3.6\times10^9$ N/m^2, $G_{13} = 6\times10^9$ N/m^2, $G_{12} = 6\times10^9$ N/m^2, $\nu_{23} = 0.34$, $\nu_{13} = 0.3$, and $\nu_{12} = 0.3$.

The material strengths of the face sheets are as follows: $X_T = 2.17\times10^9$ N/m^2, $X_C = 1.72\times10^9$ N/m^2, $Y_T = 53.8\times10^6$ N/m^2, $Y_C = 205.5\times10^6$ N/m^2, $Z_T = 53.8\times10^6$ N/m^2, $Z_C = 205.5\times10^6$ N/m^2, $S_{23} = 89.3\times10^6$ N/m^2, $S_{13} = 120.7\times10^6$ N/m^2, and $S_{12} = 120.7\times10^6$ N/m^2, where X_T, Y_T, Z_T, X_C, Y_C, and Z_C are the material strengths in tension and compression along the 1, 2, and 3 directions and S_{23}, S_{13}, and S_{12} are the shear strengths in the 23, 13, and 12 planes.

The elastic constants of the core are as follows: $E_1 = 80.4\times10^6$ N/m^2, $E_2 = 80.4\times10^6$ N/m^2, $E_3 = 1005\times10^6$ N/m^2, $G_{23} = 75.8\times10^9$ N/m^2, $G_{13} = 120.6\times10^6$ N/m^2, $G_{12} = 32.2\times10^6$ N/m^2, $\nu_{23} = 0.02$, $\nu_{13} = 0.02$, and $\nu_{12} = 0.25$.

The material strengths of the core are as follows: $Z_C = 3.83\times10^6$ N/m^2, $S_{23} = 142.3\times10^6$ N/m^2, and $S_{13} = 177.9\times10^6$ N/m^2.

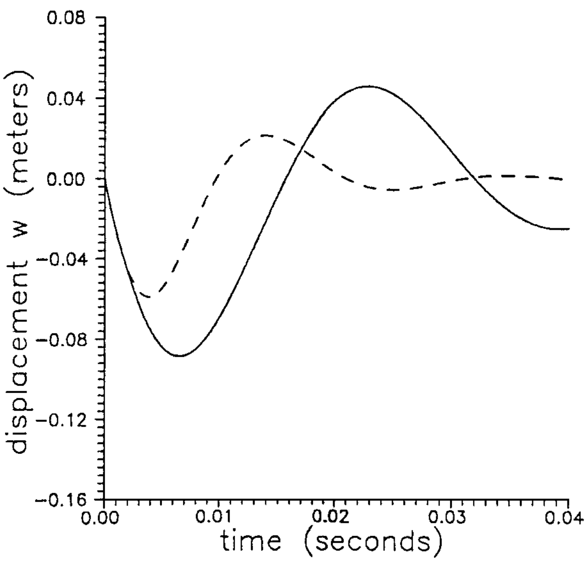

Fig. 3 Transverse displacement w (at $x = L/2$, $z = -h/2$) as a function of time in a sandwich plate dropped on elastic foundation with velocity − 30 m/s. The foundation modulus is 6.7864 × 10^7 Pa/m (sand); - - -, results of analysis without account of damage; and ——, with damage included.

Both face sheets have the same thickness 0.0025 m, and each of them consists of 25 plies with $\frac{0}{90}$-deg layup. The thickness of the core is 0.04 m. On the upper surface of the plate there is a rigid body of mass 500 kg, located symmetrically with respect to the middle of the plate's span, that has the length 0.2 m.

The modulus of the elastic Winkler foundation in the example problem, represented by Fig. 2, is 6.7864×10^8 Pa/m (clay). A time increment, used in numerical integration of equations of motion (72), is chosen to be 1×10^{-4} s. We consider a plate falling on the elastic foundation with velocity −30 m/s. The analysis of the response [time integration of the equations of motion (72)] in this and all subsequent example problems begins at the moment of time when the falling plate touches the elastic foundation. Figure 2 shows the transverse displacement of the lower surface of the plate as a function of time, computed with account of damage (solid line) and without account of damage (dashed line). In the analysis with the account of damage, the amplitude of vibration is higher. This is expected because the stiffness of the damaged structure is lower.

Figure 3 shows the transverse displacement of the plate falling on the foundation with the smaller modulus, 6.7864×10^7 Pa/m.

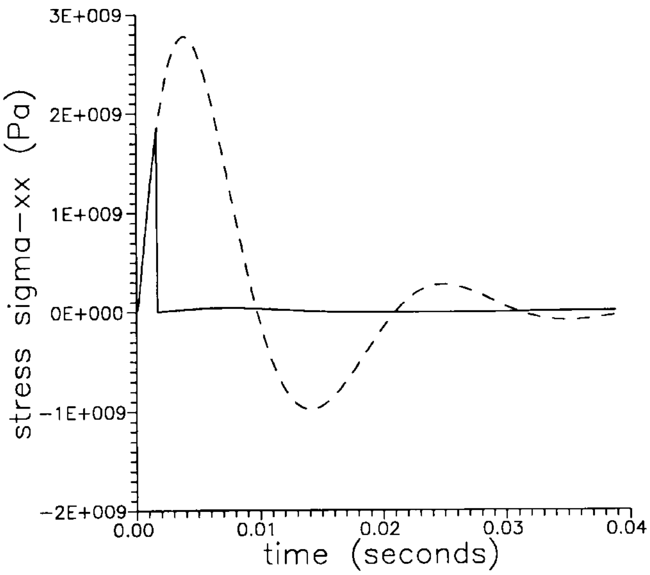

Fig. 4a Stress σ_{xx}, at $x = L/2$, $z = -h/2$, as a function of time in a sandwich plate dropped on elastic foundation with initial velocity − 30 m/s. The foundation modulus is 6.7864 × 10^7 Pa/m (sand); - - -, results of analysis without account of damage; and ——, with damage included.

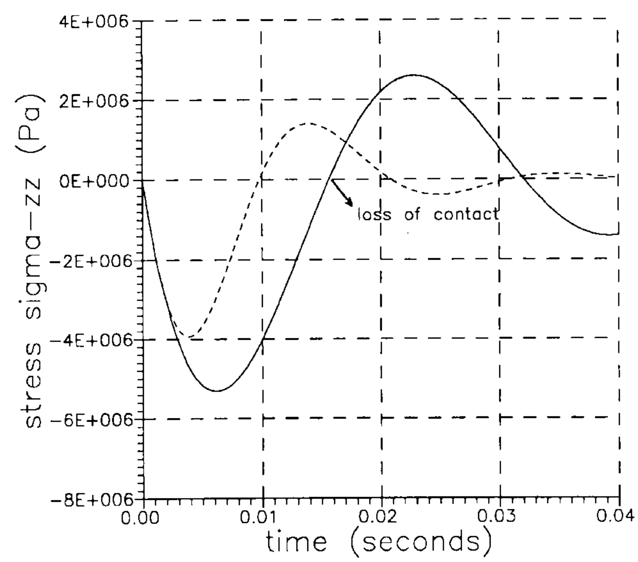

Fig. 4b Stress σ_{zz}, at $x = L/2$, $z = -h/2$, as a function of time in a sandwich plate dropped on elastic foundation with initial velocity − 30 m/s. The foundation modulus is 6.7864 × 10^7 Pa/m (sand); - - -, results of analysis without account of damage; and ——, with damage included.

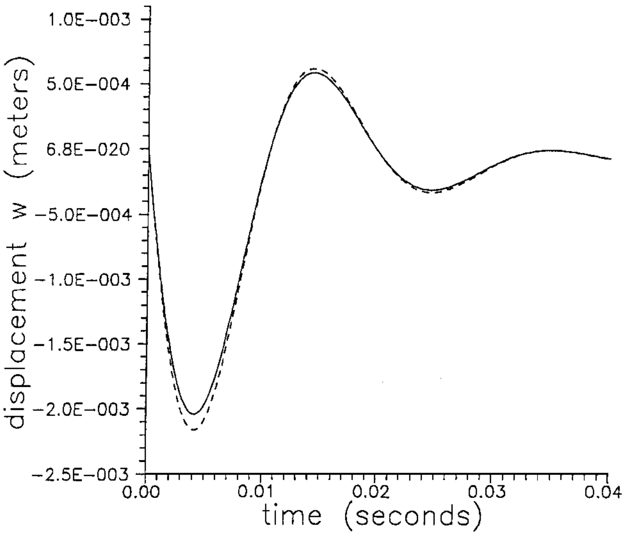

Fig. 5 Transverse displacement (at $x = -L/2$) as a function of time in a sandwich plate, with a mass on its upper surface, dropped on elastic foundation with initial velocity − 1 m/s. The foundation modulus is 6.7864×10^7 Pa/m (sand); ——, displacement of the lower surface; and - - -, displacement of the upper surface. (Under this initial velocity, damage does not occur.)

All other conditions are the same as in the preceding example. We see that, in the case of the lower modulus of the foundation, the amplitude of the transverse displacement is higher. So, the effect of the foundation stiffness is taken into account properly.

Figures 4a and 4b shows results of stress analysis with and without account of damage progression of the same sandwich plate under the impact against the elastic foundation. All conditions of the problem are the same as in the preceding example. As seen in Fig. 4a, when the fiber breakage occurs, the in-plane direct strain σ_{xx} reduces drastically, due to the degradation of the material characteristics, associated with the in-plane direction. At the moment of time $t = 0.016$ s, when stress σ_{zz} at the lower surface of the damaged plate (i.e., force of interaction between the plate and the elastic foundation per unit area) reaches a zero value (Fig. 4b), the plate loses contact with the elastic foundation and bounces up into the air. Therefore, this and all other graphs are to be considered only for the time interval during which the stress σ_{zz} at the lower surface of the plate is not positive (time interval $0 \le t \le 0.016$ s for Fig. 4b), unless the plate is glued to the elastic foundation at the moment of initial contact (i.e., is forced to stay in contact with the foundation). If the plate is forced to stay in contact with the foundation, all of the graphs are correct for any time duration.

Figure 5 shows the transverse displacement of the plate falling on the foundation with the smaller initial velocity, −1 m/s. All other conditions are the same as in the preceding example. Comparison of Figs. 3 and 5 shows that the effect of the initial velocity on the response is captured properly: the lower initial velocity causes the lower amplitude of vibration. Figure 5 shows also that the amplitude of the transverse displacement of the upper surface is higher, and this shows the capability of the model to capture the compressibility of the plate in the transverse direction.

Conclusions

The theory of the sandwich plate, presented in this paper, has a wide range of applicability. It can be used for analysis of sandwich plates with large and small thickness-to-length ratios, with thick and thin face sheets, with transversely rigid and transversely flexible face sheets and cores. The proposed finite element formulation allows one to compute accurately all stress components, both in-plane and transverse, without using finite element models with three-dimensional elements. The geometrical nonlinearity of the finite element formulation allows for a nonlinear transient analysis of a sandwich composite plate undergoing moderate rotations. The algorithm of taking account of damage progression in a dynamic problem is incorporated into the computational scheme, based on the geometrically nonlinear formulation.

Appendix: Pointwise Equilibrium Equations Variationally Consistent with the von Kármán Strain–Displacement Relations

In the equations of this Appendix, the superscripts k, which denote the numbers of the sublaminates, will not be used. These equations have a very general character, and their validity is not limited to the layerwise plate theory presented in this paper.

To derive pointwise equations of motion, consistent with the von Kármán[13] strain–displacement relations, let us substitute variations of the von Kármán strain–displacement relations, written in indicial notations,

$$\varepsilon_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha} + u_{3,\alpha}u_{3,\beta}), \qquad (\alpha, \beta = 1, 2) \quad \text{(A1)}$$

$$\varepsilon_{i3} = \tfrac{1}{2}(u_{i,3} + u_{3,i}), \qquad (i = 1, 2, 3) \quad \text{(A2)}$$

into the virtual work principle

$$\iiint_{(V)} \sigma_{ij}\delta\varepsilon_{ij}\,\mathrm{d}V = \iiint_{(V)} (\bar{F}_i - \rho\ddot{u}_i)\delta u_i\,\mathrm{d}V + \iint_{(S)} \bar{t}_i\delta u_i\,\mathrm{d}S \quad \text{(A3)}$$

where $\bar{F}_i$ are components of the body force per unit volume, ρ is density, and $\bar{t}_i$ are components of the surface traction. The variations of these strains of Eqs. (A1) and (A2) have the form

$$\delta\varepsilon_{\alpha\beta} = \tfrac{1}{2}(\delta u_{\alpha,\beta} + \delta u_{\beta,\alpha} + u_{3,\alpha}\delta u_{3,\beta} + u_{3,\beta}\delta u_{3,\alpha}) \qquad (\alpha = 1, 2; \beta = 1, 2) \quad \text{(A4)}$$

$$\delta\varepsilon_{i3} = \tfrac{1}{2}(\delta u_{i,3} + \delta u_{3,i}) \qquad (i = 1, 2, 3) \quad \text{(A5)}$$

Expression $\sigma_{ij}\delta\varepsilon_{ij}$ can be presented in the form

$$\sigma_{ij}\delta\varepsilon_{ij} = \sigma_{\alpha\beta}\delta\varepsilon_{\alpha\beta} + 2\sigma_{\alpha3}\,\delta\varepsilon_{\alpha3} + \sigma_{33}\delta\varepsilon_{33} \qquad (\alpha = 1, 2; \beta = 1, 2; i = 1, 2, 3; j = 1, 2, 3) \quad \text{(A6)}$$

When we substitute Eqs. (A4) and (A5) into Eq. (A6), we get

$$\sigma_{ij}\delta\varepsilon_{ij} = \sigma_{ij}\delta u_{i,j} + \sigma_{\alpha\beta}u_{3,\alpha}\delta u_{3,\beta} \qquad (\alpha = 1, 2; \beta = 1, 2; i = 1, 2, 3; j = 1, 2, 3) \quad \text{(A7)}$$

If one substitutes expression (A7) into the left-hand side of Eq. (A3), one gets

$$\iiint_{(V)} \sigma_{ij}\delta\varepsilon_{ij}\,\mathrm{d}V = \iint_{(S)} [\sigma_{\alpha j}n_j\delta u_\alpha + (\sigma_{3j}n_j + \sigma_{\alpha\beta}u_{3,\alpha}n_\beta)\delta u_3]\,\mathrm{d}S - \iiint_{(V)} \{\sigma_{\alpha j,j}\delta u_\alpha + [\sigma_{3j,j} + (\sigma_{\alpha\beta}u_{3,\alpha})_{,\beta}]\delta u_3\}\,\mathrm{d}V$$

$$(\alpha = 1, 2; \beta = 1, 2; i = 1, 2, 3; j = 1, 2, 3) \quad \text{(A8)}$$

where n_1, n_2, and n_3 are components of the outward unit normal vector to the surface. The substitution of expression (A8) into the virtual work principle (A3) yields

$$0 = \iiint_{(V)} \sigma_{ij}\delta\varepsilon_{ij}\,\mathrm{d}V - \iiint_{(V)} (\bar{F}_i - \rho\ddot{u}_i)\delta u_i\,\mathrm{d}V - \iint_{(S)} \bar{t}_i\delta u_i\,\mathrm{d}S$$

$$= \iint_{(S)} [(\sigma_{\alpha j}n_j - \bar{t}_\alpha)\delta u_\alpha\,\mathrm{d}S + (\sigma_{3j}n_j + \sigma_{\alpha\beta}u_{3,\alpha}n_\beta - \bar{t}_3)\delta u_3]\,\mathrm{d}S - \iiint_{(V)} \{(\sigma_{\alpha j,j} + \bar{F}_\alpha - \rho\ddot{u}_\alpha)\delta u_\alpha + [\sigma_{3j,j} + (\sigma_{\alpha\beta}u_{3,\alpha})_{,\beta} + \bar{F}_3 - \rho\ddot{u}_3]\delta u_3\}\,\mathrm{d}V \qquad (\alpha = 1, 2; \beta = 1, 2; j = 1, 2, 3) \quad \text{(A9)}$$

If one equates to zero the coefficients of variations of displacements, one obtains the equations of motion

$$\sigma_{\alpha j,j} + \bar{F}_\alpha = \rho \ddot{u}_\alpha, \qquad \sigma_{3j,j} + (\sigma_{\alpha\beta} u_{3,\alpha})_{,\beta} + \bar{F}_3 = \rho \ddot{u}_3$$

$$(\alpha = 1, 2;\ \beta = 1, 2;\ j = 1, 2, 3) \quad \text{(A10)}$$

and natural boundary conditions

$$\sigma_{\alpha j} n_j = \bar{t}_\alpha, \qquad \sigma_{3j} n_j + \sigma_{\alpha\beta} u_{3,\alpha} n_\beta = \bar{t}_3 \quad \text{at} \quad S_\sigma$$

$$(\alpha = 1, 2;\ \beta = 1, 2;\ j = 1, 2, 3) \quad \text{(A11)}$$

where S_σ is a part of the surface on which displacement constraints are not imposed. Equations of motion (A12) in expanded form are

$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + \bar{F}_x = \rho \ddot{u} \quad \text{(A13)}$$

$$\sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + \bar{F}_y = \rho \ddot{v} \quad \text{(A14)}$$

$$\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + \frac{\partial}{\partial x}(\sigma_{xx} w_{,x} + \sigma_{yx} w_{,y}) + \frac{\partial}{\partial y}(\sigma_{xy} w_{,x} + \sigma_{yy} w_{,y}) + \bar{F}_z = \rho \ddot{w} \quad \text{(A15)}$$

The boundary conditions (A11) in expanded form are

$$\sigma_{xx} n_x + \sigma_{xy} n_y + \sigma_{xz} n_z = \bar{t}_x \quad \text{(A16)}$$

$$\sigma_{yx} n_x + \sigma_{yy} n_y + \sigma_{yz} n_z = \bar{t}_y \quad \text{(A17)}$$

$$\sigma_{zx} n_x + \sigma_{zy} n_y + \sigma_{zz} n_z + \sigma_{xx} w_{,x} n_x + \sigma_{yy} w_{,y} n_y + \sigma_{xy}(w_{,x} n_y + w_{,y} n_x) = \bar{t}_z \quad \text{(A18)}$$

In the postprocessing stage of the finite element analysis, the computation of the transverse stresses is done with the use of the pointwise equations of motion (A13–A15), variationally consistent with the von Kármán[13] strain-displacement relations (2–7).

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